CONIC SECTION

1. A parabola of latus rectum $4a$, touches a fixed equal parabola, the axes of the two curves being parallel; prove that the locus of the vertex of the moving curve is a parabola of latus rectum $8a$.

Sol:

1. Let fixed parabola is $y^2 = 4ax$ the equal parabola (variable) having parabola axis can be taken as $(y-k)^2 = \pm 4a(x-h)$ where (h,k) is vertex.

$\Rightarrow$ Both parabola touches each other if at common point (x_1, y_1) both have same slopes

$$\Rightarrow \left(\frac{dy}{dx}\right)C_1 = \left(\frac{dy}{dx}\right)C_2 \text{ at } (x_1y_1) \Rightarrow \frac{2a}{y_1} = \pm\frac{2a}{y_1-k}$$

$$\Rightarrow y_1 = \frac{k}{2} \text{ hence putting this in } (y-k)^2 = -4a(x-h)$$

Now (x_1, y_1) lies on $y^2 = 4ax \Rightarrow \frac{k^2}{4} = 4a\left(h - \frac{k^2}{16a}\right) \Rightarrow k^2 = 8ah$

$\Rightarrow$ Locus of vertex is $y^2 = 8ax \Rightarrow$ Latus rectum $= 8a$

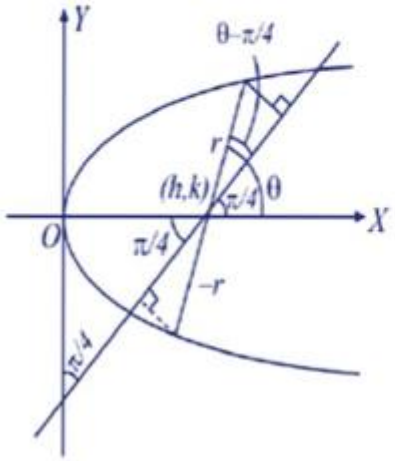
2. A family of chords of the parabola $y^2 = 4ax$ is drawn so that their projections on a straight line inclined equally to both the axes are all of a constant length c ; prove that the locus of their middle point is the curve $(y^2 - 4ax)(y + 2a)^2 - 2a^2c^2 = 0$.

Sol:

2. Let the equation of straight line (h,k) as its mid point, then

$$\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r \qquad (1)$$

Any point on the line (1) is $(h+r\cos\theta, k+r\sin\theta)$



Solving with the equation of parabola.

$y^2 = 4ax$, we get

$$(k+r\sin\theta)^2 = 4a(h+r\cos\theta)$$

$$\Rightarrow r^2\sin^2\theta + 2r(k\sin\theta - 2a\cos\theta) + k^2 - 4ah = 0 \qquad (2)$$

which is quadratic in r

.

This roots of the quadratic equation will be equal but of opposite sign as (h,k) is the mid point.

$$\text{Sum of roots} = -\frac{2(k\sin\theta - 2a\cos\theta)}{\sin^2\theta} = 0$$

$$\therefore k\sin\theta - 2a\cos\theta = 0$$

$$\therefore \tan\theta = \frac{2a}{k}$$

Now from (2), $r^2\sin^2\theta + (k^2 - 4ah) = 0$

$$\Rightarrow r^2 \cdot \frac{4a^2}{(4a^2+k^2)} + (k^2 - 4ah) = 0 \qquad (3)$$

Length of the chord will be $2r$. Angle between the two lines will

be $(\theta - \pi/4)$ and the projection of the chord on the given line will be $2r\cos(\theta - \pi/4) = c$

$$\Rightarrow \frac{2r}{\sqrt{2}}(\cos\theta + \sin\theta) = c \Rightarrow \frac{2r}{\sqrt{2}}\left(\frac{k+2a}{\sqrt{4a^2+k^2}}\right) = c$$

$$\Rightarrow 2r^2(k+2a)^2 = c^2(4a^2+k^2) \Rightarrow \frac{2r^2}{(4a^2+k^2)} = \frac{c^2}{(k+2a)^2} \qquad (4)$$

From (3) and (4), we get

$$\frac{2a^2c^2}{(k+2a)^2} + (k^2 - 4ah) = 0$$

$$\Rightarrow (k^2 - 4ah)(k+2a)^2 + 2a^2c^2 = 0$$

Hence the locus of the middle point is

$$(y^2 - 4ax)(y+2a)^2 + 2a^2c^2 = 0$$

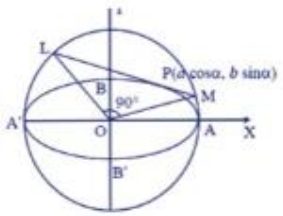
3. The tangent at the point α on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the auxiliary circle in two points which subtend a right angle at the centre. The eccentricity of the ellipse is $\left(\frac{\text{cosec}\alpha}{\sqrt{\lambda + \cot^2\alpha}}\right)$ then λ equal to ____.

Sol:

3. (2) Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

...(i)

Its auxiliary circle is $x^2 + y^2 = a^2$



...(ii)

Let $P \equiv (a\cos\alpha\, b\sin\alpha)$

Equation of tangent to the ellipse at $P(a\cos\alpha\, b\sin\alpha)$ is

$$\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b} = 1 \qquad (iii)$$

Making equation (ii) homogeneous with the help of (iii), we get :

$$x^2+y^2-a^2\left(\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}\right)^2=0$$

or

$$(1-\cos^2\alpha)x^2+\left(1-\frac{a^2}{b^2}\sin^2\alpha\right)y^2-2\frac{a}{b}\cos\alpha\sin\alpha xy=0 \qquad \text{(iv)}$$

(iv) is the joint equation of OL and OM. Since $\angle LOM=90^\circ$

$\therefore$

Coefficient of x^2 + coefficient of $y^2=0 \therefore 1-\cos^2\alpha+1-\frac{a^2}{b^2}\sin^2\alpha=0$

or

$$\sin^2\alpha\left(\frac{a^2}{b^2}-1\right)=1 \text{ or } \sin^2\alpha\left(\frac{1}{1-e^2}-1\right)-1\left[\because b^2=a^2(1-e^2)\right]$$

or

$$e^2\sin^2\alpha=1-e^2 \text{ or } e^2(1+\sin^2\alpha)=1 \text{ or}$$

$$e=\frac{1}{\sqrt{1+\sin^2\alpha}}=\frac{\operatorname{cosec}\alpha}{\sqrt{2+\cot^2\alpha}}$$