INTEGRAL CALCULUS

1. $\int_0^{\infty}\left[\frac{2}{e^x}\right]dx+\int_{-1/2}^{1/2}\sqrt{\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2-2}dx$ is equal to $\ln\left(\frac{2^a}{3^b}\right)$ then $(a-b)$ is equal to ____ ($[x]$ = G.I.F)

Sol:

1. (5) Let

$$I_1=\int_0^{\infty}\left[\frac{2}{e^x}\right]dx=\int_0^{\ln 2}\left[\frac{2}{e^x}\right]dx+\int_{\ln 2}^{\infty}\left[\frac{2}{e^x}\right]dx=\int_0^{\ln 2}1dx+\int_{\ln 2}^{\infty}0dx=\ln 2$$

$$\text{Let } I_2=\int_{-1/2}^{1/2}\sqrt{\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2-2}=\int_{-1/2}^{1/2}\sqrt{\left(\frac{x+1}{x-1}-\frac{x-1}{x+1}\right)^2}dx$$

$$=\int_{-1/2}^{1/2}\left|\frac{4x}{x^2-1}\right|dx=4\int_0^{1/2}\frac{2|x|}{|x^2-1|}dx=-4\int_0^{1/2}\frac{2x}{x^2-1}dx=$$

$$-4\left[\ln|x^2-1|\right]_0^{1/2}=4\ln\left(\frac{4}{3}\right)$$

$$I_1+I_2=\ln 2+4(2\ln 2-\ln 3)=9\ln 2-4\ln 3=\ln\left(\frac{2^9}{3^4}\right)$$

$$\text{Now, } a-b=9-4=5$$

2. $\lim_{n\to\infty}\left(\frac{^{3n}C_n}{^{2n}C_n}\right)^{1/n}$ is equal to a/b, where a and b are co-prime number then $a+b$ is equal to ____

2. (43) $$y=\lim_{n\to\infty}\left(\frac{\frac{3n!}{2n!\cdot n!}}{\frac{2n!}{n!\cdot n!}}\right)^{1/n}$$

$$y=\lim_{n\to\infty}\left(\frac{(3n)!n!}{(2n)!(2n)!}\right)^{1/n}=\lim_{n\to\infty}\left(\frac{\{(3n)(3n-1)\ldots(2n+1)\}}{\{(2n)(2n-1)\ldots.(n+1)\}}\right)^{1/n}$$

$$=\lim_{n\to\infty}\left(\prod_{r=0}^{n-1}\frac{3n-r}{2n-r}\right)^{1/n}$$

$$\ln y=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\ln\left(\frac{3n-r}{2n-r}\right)$$

$$\ln y=\int_0^1\ln\left(\frac{3-x}{2-x}\right)dx$$

$$\ln y=\int_0^1\ln(3-x)dx-\int_0^1\ln(2-x)dx$$

$$\ln y = [(x+2)\ln(x+2) - (x+2)]_0^1 - [(x+1)\ln(x+1) - (x+1)]_0^1$$
$$\ln y = [(3\ln 3 - 3) - (2\ln 2 - 2)] - [(2\ln 2 - 2) - (1 \cdot \ln 1 - 1)]$$

3. If $I_1 = \int_0^\infty f(x^n + x^{-n}) \ln x \cdot \frac{dx}{x}$ & $I_2 = \int_0^\infty f(x^n + x^{-n}) \ln x \cdot \frac{dx}{(1+x^2)}$ then :
(A) $I_1 = 0$ (B) $I_2 \neq 0$ (C) $I_1 + I_2 = 0$ (D) $2I_1 + 3I_2 = 0$

3. (ACD)

Let

$$t = \ln x \Rightarrow x = e^t$$
$$dx = e^t dt$$

$$I_1 = \int_0^\infty A(x^n + x^{-n})(\ln x)\frac{dx}{x} = \int_{-\infty}^\infty A(e^{nt} + e^{-nt})\, t\, dt = 0$$

Now;

$$I_2 = \int_0^\infty A(x^n + x^{-n})(\ln x)\frac{dx}{1+x^2} = \int_{-\infty}^\infty A(e^{nt} + e^{-nt})\, t \frac{e^t}{1+e^{2t}}\, dt = 0$$

4. If $f(x) = e^x + \int_0^1 (e^x + te^{-x}) f(t)dt$ then :
(A) $f(0) = \frac{e^2 - e + 2}{4e - 2e^2}$ (B) $f(1) = \frac{2e^2 + e - 2}{4e - 2e^2}$ (C) $f(0) = \frac{e^2 + e - 2}{4e - 2e^2}$ (D) $f(0) = \frac{2e^2 - e - 1}{4e - 2e^2}$

4. (CD)

$$f(x) = e^x\left(1 + \int_0^1 f(t)dt\right) + e^{-x}\int_0^1 tf(t)dt$$

Let

$$A = 1 + \int_0^1 f(t)dt \quad B = \int_0^1 tf(t)dt.$$

So;

$$f(x) = e^x(A) + Be^{-x}$$

Now; $A = 1 + \int_0^1 (Ae^t + Be^{-t})\,dt$

$$A = 1 + A(e-1) + B(1-(1/e))$$
$$(2-e)A + ((1/e)-1)B = 1 \qquad (i)$$

Now;

$$B = \int_0^1 t(Ae^t + Be^{-t})\,dt$$

$$B = A + B\left(1 - \frac{2}{e}\right)$$
$$A - 2e^{-1}B = 0 \qquad (ii)$$

From (i) and (ii) ; $A = \frac{2(e-1)}{4e-2e^2}$; $B = \frac{e-1}{4-2e}$

Hence, $f(x) = \frac{2(e-1)}{4e-2e^2} \cdot e^x + \frac{e-1}{4-2e} e^{-x}$

$$f(0) = \frac{2(e-1)}{4e-2e^2} + \frac{e-1}{4-2e} = \frac{2e-2+e^2-e}{4e-2e^2} = \frac{e^2+e-2}{4e-2e^2}$$

$$f(1) = \frac{2(e-1)}{4e-2e^2} e + \frac{e-1}{4-2e}(e^{-1}) = \frac{2(e^2-e)}{4e-2e^2} + \frac{e-1}{4e-2e^2}$$

$$f(1) = \frac{2e^2-e-1}{4e-2e^2}$$